Abstract Algebra II
Exercise Set 2: Group Actions and Sylow

Group actions

Question 1. For the given groups and sets, show that α is a group action:

- $G = (\mathbb{Z}, +)$, $X = \{1, 2, 3\}$, and $\alpha : G \times X \to X : (g, x) = 1 \cdot x$;
- $G = S_4$, $X = S_4$, and $\alpha : G \times X \to X : (g, x) = gx$;
- $G = S_4$, $X = S_4$, and $\alpha : G \times X \to X : (g, x) = xg$;
- $G = S_n$, for any $n > 2$, $X = \{0, 1\}$, and
$$\alpha : G \times X \to X : (g, x) = \begin{cases} x, & g \text{ even,} \\ x + 1 \mod 2, & g \text{ odd} \end{cases}$$
($0 + 1 \equiv 1 \mod 2$ and $1 + 1 \equiv 0 \mod 2$, so even permutations preserve x and odd permutations swap 1 and 2);
- $G = D_{2 \cdot 5}$, $X = G$, $\alpha : G \times X \to X : (g, x) = gxg^{-1}$;
- $G = D_{2 \cdot 5}$, $X = G$, $\alpha : G \times X \to X : (g, x) = gxg^{-1}$;

Question 2. Find all orbits of the following group actions:

- $G = (\mathbb{Z}, +)$, $X = \{1, 2, 3\}$, and $\alpha : G \times X \to X : (g, x) = 1 \cdot x$;
- $G = S_4$, $X = S_4$, and $\alpha : G \times X \to X : (g, x) = gx$;
- $G = S_n$, for any $n > 2$, $X = \{0, 1\}$, and
$$\alpha : G \times X \to X : (g, x) = \begin{cases} x, & g \text{ even,} \\ x + 1 \mod 2, & g \text{ odd.} \end{cases}$$
- $G = S_4$, $X = \{A, B, C, D\}$, and $\alpha : G \times X \to X : (g, x) = 1 \cdot g(x)$ (for example, if $g = (A, B, C)$ and $x = B$ then $g(x) = C$, as g sends B to C);
- $G = \mathbb{Z}_16 \times \mathbb{Z}_12$, $X = G$, and $\alpha : G \times X \to X : (g, x) = gxg^{-1}$.

Question 3. Find the stabiliser of the following elements under the group actions:

- Stab(2), where $G = (\mathbb{Z}, +)$, $X = \{1, 2, 3\}$, and $\alpha : G \times X \to X : (g, x) = 1 \cdot x$;
- Stab$\big((12)(34)\big)$, where $G = S_4$, $X = S_4$, and $\alpha : G \times X \to X : (g, x) = gx$;
- Stab(1), where $G = S_n$, for any $n > 2$, $X = \{0, 1\}$, and
$$\alpha : G \times X \to X : (g, x) = \begin{cases} x, & g \text{ even,} \\ x + 1 \mod 2, & g \text{ odd.} \end{cases}$$
- Stab(C), where $G = S_4$, $X = \{A, B, C, D\}$, and $\alpha : G \times X \to X : (g, x) = 1 \cdot g(x)$;
- Stab$\big((3, 5)\big)$, where$G = \mathbb{Z}_16 \times \mathbb{Z}_12$, $X = G$, and $\alpha : G \times X \to X : (g, x) = gxg^{-1}$.

Question 4. For the previous two questions verify the orbit stabiliser theorem. That is, for the stabiliser of an element x, consider the corresponding orbit and compare the sizes of the orbit, stabiliser, and group.

Question 5. Determine how many p-Sylow subgroups there are in the following groups:

- $G = \mathbb{Z}_4$;
- $G = \mathbb{Z}_6$;
- $G = \mathbb{Z}_{12}$;
- $G = V_4$, the Klein-4 group;
- $G = S_3$;
- $G = D_{2\cdot 3}$, the dihedral group of a regular triangle;
- $G = D_{2\cdot 4}$, the dihedral group of a square.

Question 6. Find all the p-Sylow subgroups of the following groups:

- $G = \mathbb{Z}_5$;
- $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein-4 group);
- $G = D_{2\cdot 5}$;
- $G = A_4$.

Question 7. Find the conjugacy classes of all p-Sylow subgroups of the following groups:

- $G = \mathbb{Z}_5$;
- $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein-4 group);
- $G = D_{2\cdot 5}$;
- $G = A_4$.